Seminar Lecture - Propositional Calculus

1. **Definition:** A **proposition** is a statement that is either TRUE or FALSE.

    - $1 + 2 = 3$
    - All functions are continuous
    - What's up?
    - $x = 5$. ('open sentence' - NOT a proposition)
    - 'This sentence is false.'

2. Operations on propositions:

    (a) Negation: $\neg P$: '*not* P'; opposite truth value of P.
    (b) Conjunction: $P \wedge Q$: 'P *and* Q'; True only when BOTH are true; false otherwise.
    (c) Disjunction: $P \vee Q$: 'P *or* Q'; False only when BOTH are false; true otherwise.

3. **Truth Tables:** Arrays which summarize all truth values of a propositional form:

    - **Negation:**

P	$\neg P$
T	F
F	T

    - **Conjunction:**

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

    - **Disjunction:**

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

4. **Order of Operations:** $\neg$ and $\wedge$ apply to the 'closest' propositions and are evaluated first; next is $\vee$: $\neg P \vee Q \wedge R$ means $(\neg P) \vee (Q \wedge R)$.

5. A **tautology** is something that is always **true**.

6. A **contradiction** is something that is always **false**.

7. **Definition:** Two propositional forms are called **logically equivalent** if they have the same truth tables. If P and Q are logically equivalent, we write $P \equiv Q$.

8. **Theorem:** 'The Propositional Calculus':

   (a) $\neg(\neg P) \equiv P$ (Idempotency of $\neg$)
   (b) $P \wedge Q \equiv Q \wedge P$ (Commutativity of $\wedge$)
   (c) $P \vee Q \equiv Q \vee P$ (Commutativity of $\vee$)
   (d) $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$ (Associativity of $\wedge$)
   (e) $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$ (Associativity of $\vee$)
   (f) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ (Distributive Law: $\wedge$ over $\vee$)
   (g) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ (Distributive Law: $\vee$ over $\wedge$)
   (h) $P \wedge (\neg P) \equiv F$ (No double standards!)
   (i) $P \vee (\neg P) \equiv T$ (Law of the Excluded Middle)
   (j) $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$ (DeMorgan's Law)
   (k) $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$ (DeMorgan's Law)
   (l) $P \wedge T \equiv P$ (Identity of $\wedge$)
   (m) $P \vee F \equiv P$ (Identity of $\vee$)
   (n) $P \wedge F \equiv F$ (Absorption of F with $\wedge$)
   (o) $P \vee T \equiv T$ (Absorption of T with $\vee$)

9. **Example:** $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ (Distributive Law: $\vee$ over $\wedge$)

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

10. We can use the Propositional Calculus to verify equivalences without resorting to a truth table!

11. **Example:** Show $(P \vee Q) \vee [(\neg P) \wedge (\neg Q)]$ is a tautology.

   (a) **Method 1:** Construct a truth table for $(P \vee Q) \vee [(\neg P) \wedge (\neg Q)]$ and show it is always true.
   (b) **Method 2:** Use the Propositional Calculus!

$$\begin{aligned} (P \vee Q) \vee [(\neg P) \wedge (\neg Q)] &\equiv (P \vee Q) \vee [\neg(P \vee Q)] && \text{DeMorgan's Law} \\ &\equiv T && \text{Law of the Excluded Middle applied to } (P \vee Q) \end{aligned}$$

12. **Definition:** The **conditional**, notated, $P \implies Q$ is the propositional form $(\neg P) \vee Q$.

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

13. **Notes:**

- P is called the 'hypothesis' or 'antecedent'
- Q is called the 'conclusion' or 'consequent'
- $P \implies Q$ is only false when P is true and Q is false; otherwise it's true.
- $P \implies Q$ may be translated as:
  (a) P implies Q.
  (b) If P, then Q.
  (c) Q, if P.
  (d) P only if Q.
  (e) P is sufficient for Q.
  (f) Q is necessary for P.
  (g) Q whenever P.
- Think of 'If P, then Q' as a promise: the promise is only broken if P is true, but Q is false.

14. Given a conditional $P \implies Q$:

  (a) $Q \implies P$ is called the **converse**
  (b) $(\neg P) \implies (\neg Q)$ is called the **inverse**
  (c) $(\neg Q) \implies (\neg P)$ is called the **contrapositive**

15. **THEOREM:** $[P \implies Q] \equiv [(\neg Q) \implies (\neg P)]$

16. Hence, a conditional is logically equivalent to its contrapositive.

17. **NOTE:** A conditional is **NOT** logically equivalent to its converse.

18. **NOTE:** Since the inverse of a conditional is the contrapositive of its converse, the inverse of a conditional is logically equivalent to the converse of the conditional.

19. **Theorem:** Properties of $\implies$:

  (a) $$\left.\begin{array}{l}(P \implies Q) \implies [(P \wedge R) \implies (Q \wedge R)] \\ (P \implies Q) \implies [(P \vee R) \implies (Q \vee R)]\end{array}\right\} \text{Addition Properties}$$
  (b) $[(P \implies Q) \wedge (Q \implies R)] \implies (P \implies R)$ (Transitivity)

20. **Definition:** The **biconditional**, notated, $P \iff Q$ is the equivalent to: $(P \implies Q) \wedge (Q \implies P)$.

P	Q	$P \iff Q$
T	T	T
T	F	F
F	T	F
F	F	T

21. $P \iff Q$ is true precisely when P and Q have the same truth values : '$\iff$' is '$\equiv$'

22. $P \iff Q$ may be translated as:

    (a) P if and only if Q.
    (b) P iff Q.
    (c) P is necessary and sufficient for Q.

23. **Theorem:** Properties of $\iff$:

    (a) $\left.\begin{array}{l} (P \iff Q) \implies [(P \wedge R) \iff (Q \wedge R)] \\ (P \iff Q) \implies [(P \vee R) \iff (Q \vee R)] \end{array}\right\}$ Addition Properties
    (b) $[(P \iff Q) \wedge (Q \iff R)] \implies (P \iff R)$ (Transitivity)

Seminar Lecture - Predicate Calculus

1. An **open sentence** is a 'proposition with a variable.' Open sentences are sometimes called *predicates*.
2. **Example:** $x = 2$; x is an elephant; etc.
3. Generic proposition: P; generic open sentence: $P(x)$.
4. Quantifiers:
   - $\forall$: 'for all.'
   - $\exists$: 'there exists.'
   - $\exists!$: 'there exists a unique.'
5. **Examples:**
   (a) 'All elephants are grey.' may be symbolized as: '$\forall\, x : x$ is an elephant $\implies$ x is grey.'
   (b) 'Some elephants are grey.' may be symbolized as: '$\exists\, x : x$ is an elephant $\wedge\, x$ is grey.'
   (c) 'There is only one grey elephant.' may be symbolized as: $\exists!\, x : x$ is an elephant $\wedge\, x$ is grey.'
6. Negation of quantified sentences:
   - $\neg\,(\forall\, x : P(x)) \iff (\exists\, x : \neg P(x))$
   - $\neg\,(\exists\, x : P(x)) \iff (\forall\, x : \neg P(x))$
   - $\neg\,(\exists!\, x : P(x)) \iff [(\forall\, x : \neg P(x)) \vee (\exists\, x, y, x \neq y : P(x) \wedge P(y))]$
7. **Examples:**
   - The negation of: 'All elephants are grey.' is: '*Some* elephants are *not* grey.'
   - The negation of: 'Some elephants are grey.' is: '*All* elephants are *not* grey.'
   - The negation of: 'There is only one grey elephant.' is: 'Either *all* elephants are *not* grey, *or* there is *more than one* grey elephant.'
8. Other symbols commonly used:
   - $\in$: 'is an element of' (used to show membership in a set)
   - $\ni$: 'such that.'
   - $\mathbb{N} = \{1, 2, 3, ...\}$ a.k.a. 'The natural numbers.'
   - $\mathbb{Z} = \{0, 1, -1, 2, -2, ...\}$ a.k.a. 'The integers.'
   - $\mathbb{Q} = \left\{\dfrac{x}{y} : x \in \mathbb{Z} \wedge y \in \mathbb{N}\right\}$ a.k.a. 'The rational numbers.'
   - $\mathbb{R}$ a.k.a. 'The real numbers.'
9. Order of quantifiers:
   - '$\forall\, x\, \exists\, y$' permits the choice of y to depend on the choice of x.
   - '$\exists\, y\, \forall\, x$' requires the choice of y work regardless of the choice of x.
10. **Examples:**
    - $\forall\, x \in \mathbb{R}\ \exists\, y \in \mathbb{R} \ni x + y = 0$ is **TRUE**: given an x, you can choose $y = -x$.
    - $\exists\, y \in \mathbb{R}\ \forall\, x \in \mathbb{R} \ni x + y = 0$ is **FALSE**: because there isn't one y that works for all x.
11. **IMPORTANT EXAMPLE:** $\lim_{x \to a} f(x) = L$ means:

$$\forall\, \epsilon > 0\, \exists\, \delta > 0 \ni 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$

Seminar Lecture - Set Theory

1. **Definition:** A **set** is a well-defined collection of objects meaning that it is possible to determine if an object belongs to the set or not, without prejudice.

2. The **empty set**, denoted $\emptyset$, is defined as the set containing no elements.

3. If A denotes a set and x denotes an object, or **element**, belonging to A, we write $x \in A$. If x does not belong to A, we write $x \notin A$.

4. The **empty set**, denoted $\emptyset$, is defined as the set containing no elements.

5. **Definition:** A **universal set** is a set containing all of the objects under consideration and will be denoted X. (There are fundamental issues with making such a set 'too big.' - see Russel's Paradox.)

6. If A and B are two sets, we say 'A is **equal** to B' and write $A = B$ if and only if A and B contain exactly the same elements. In symbols:

$$A = B \iff (\forall x : x \in A \iff x \in B)$$

   If every element of A is an element of B, then we say 'A is a **subset** of B' (B is a **superset** of A') and write $A \subseteq B$ ('$B \supseteq A$). In symbols:

$$A \subseteq B \iff (\forall x : x \in A \implies x \in B)$$

7. Hence, when speaking of sets, A, and B, we assume they are both subsets of a universal set, X.

8. **Definitions:** Let A and B be subsets of a universal set, X.

   (a) $\tilde{A} = \{x : x \notin A\}$ is called the *complement* of A.
   (b) $A \cap B = \{x : (x \in A) \wedge (x \in B)\}$ is called the *intersection* of A with B.
   (c) If $A \cap B = \emptyset$, we say A and B are *disjoint* sets.
   (d) $A \cup B = \{x : (x \in A) \vee (x \in B)\}$ is called the *union* of A with B.
   (e) If A and B are known to be disjoint, we write $A \uplus B$ to signify $A \cup B$.

9. **Definition:** Given a set X, the *power set* of X, $\mathcal{P}(X)$ is: $\mathcal{P}(X) = \{A : A \subseteq X\}$. (i.e., $\mathcal{P}(X)$ is the set of all subsets of X.)

10. **VENN DIAGRAMS:** A way to 'visualize' sets.

11. **Theorem:** 'The Set Calculus': Let A, B, and C be subsets of a universal set, X.

    (a) $\tilde{\tilde{A}} = A$ (Idempotency of $\tilde{}$)

    (b) $A \cap B = B \cap A$ (Commutativity of $\cap$)

    (c) $A \cup B = B \cup A$ (Commutativity of $\cup$)

    (d) $(A \cap B) \cap C = A \cap (B \cap C)$ (Associativity of $\cap$)

    (e) $(A \cup B) \cup C = A \cup (B \cup C)$ (Associativity of $\cup$)

    (f) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive Law: $\cap$ over $\cup$)

    (g) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributive Law: $\cup$ over $\cap$)

    (h) $\left.\begin{array}{l} A \cap \tilde{A} = \emptyset \\ A \cup \tilde{A} = X \end{array}\right\} \quad A \mathbin{\dot\cup} \tilde{A} = X$

    (i) $\left.\begin{array}{l} \widetilde{A \cap B} = \tilde{A} \cup \tilde{B} \\ \widetilde{A \cup B} = \tilde{A} \cap \tilde{B} \end{array}\right\}$ DeMorgan's Laws

    (j) $A \cap X = A$ (Identity of $\cap$)

    (k) $A \cup \emptyset = A$ (Identity of $\cup$)

    (l) $A \cap \emptyset = \emptyset$ (Absorption of $\emptyset$ with $\cap$)

    (m) $A \cup X = X$ (Absorption of X with $\cup$)

    (n) $\tilde{X} = \emptyset$

    (o) $\tilde{\emptyset} = X$

12. **Theorem:** Properties of $\subseteq$: Let A, B, and C be subsets of a universal set, X.

    (a) $\emptyset \subseteq A$

    (b) $A \subseteq A$ (Reflexive Property)

    (c) $A = B \iff (A \subseteq B) \wedge (B \subseteq A)$

    (d) $(A \subseteq B) \wedge (B \subseteq C) \implies A \subseteq C$ (Transitive Property)

    (e) $A \cap B \subseteq A$

    (f) $A \cap B = A \iff A \subseteq B$

    (g) $A \subseteq A \cup B$

    (h) $A = A \cup B \iff B \subseteq A$

    (i) $A \subseteq B \implies A \cap C \subseteq B \cap C$

    (j) $A \subseteq B \implies A \cup C \subseteq B \cup C$

13. **Definition:** $A - B = \{x : x \in A \wedge x \notin B\}$ is called the *set theoretic difference* of A and B.

14. **Theorem:** Properties of $-$: Let A and B be subsets of a universal set, X:

    (a) $A - B = A \cap \tilde{B}$

    (b) $X - A = \tilde{A}$

    (c) $A - \emptyset = A$

    (d) $A - (B \cap C) = (A - B) \cup (A - C)$

    (e) $A - (B \cup C) = (A - B) \cap (A - C)$

Seminar Lecture - Families of Sets

1. A *family of sets* is a set of sets. We usually denote families of sets by calligraphic letters: $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, etc.

2. **Example:** Given a set, X, $\mathcal{P}(X)$ is a family of sets.

3. $\displaystyle\bigcap_{A\in\mathcal{A}} A = \{x : \forall A \in \mathcal{A}, x \in A\}$; i.e., $\displaystyle x \in \bigcap_{A\in\mathcal{A}} A \iff \forall A \in \mathcal{A}, x \in A$

4. $\displaystyle\bigcup_{A\in\mathcal{A}} A = \{x : \exists A \in \mathcal{A} \ni x \in A\}$; i.e., $\displaystyle x \in \bigcup_{A\in\mathcal{A}} A \iff \exists A \in \mathcal{A} \ni x \in A$

5. Oftentimes, we *index* the family of sets. For example, $A_n = [-n, n]$ for $n \in \mathbb{N}$. In this case...

   - $\displaystyle\bigcap_{A\in\mathcal{A}} A = \bigcap_{n\in\mathbb{N}} A_n = \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty}[-n, n] = [-1, 1]$
   - $\displaystyle\bigcup_{A\in\mathcal{A}} A = \bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty}[-n, n] = (-\infty, \infty)$

6. In the previous example, $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$. These sets are called 'nested.'

7. In general, by an *indexed family of sets*, we mean a family of sets, $\mathcal{A}$ along with a function, $f : \Delta \twoheadrightarrow \mathcal{A}$. If $\alpha \in \Delta$, we write $f(\alpha) = A_\alpha$, and write the family: $\mathcal{A} = \{A_\alpha : \alpha \in \Delta\}$

8. If we have an indexed family of sets...

   - $\displaystyle\bigcap_{A\in\mathcal{A}} A = \bigcap_{\alpha\in\Delta} A_\alpha$
   - $\displaystyle\bigcup_{A\in\mathcal{A}} A = \bigcup_{\alpha\in\Delta} A_\alpha$

9. **Theorem:** Provided $\Delta \neq \emptyset$:

   (a) $\displaystyle B \cap \left(\bigcap_{\alpha\in\Delta} A_\alpha\right) = \bigcap_{\alpha\in\Delta} (B \cap A_\alpha)$

   (b) $\displaystyle B \cup \left(\bigcap_{\alpha\in\Delta} A_\alpha\right) = \bigcap_{\alpha\in\Delta} (B \cup A_\alpha)$

   (c) $\displaystyle B \cap \left(\bigcup_{\alpha\in\Delta} A_\alpha\right) = \bigcup_{\alpha\in\Delta} (B \cap A_\alpha)$

   (d) $\displaystyle B \cup \left(\bigcup_{\alpha\in\Delta} A_\alpha\right) = \bigcup_{\alpha\in\Delta} (B \cup A_\alpha)$

   (e) $$\left.\begin{aligned} \widetilde{\bigcap_{\alpha\in\Delta} A_\alpha} &= \bigcup_{\alpha\in\Delta} \tilde{A}_\alpha \\ \widetilde{\bigcup_{\alpha\in\Delta} A_\alpha} &= \bigcap_{\alpha\in\Delta} \tilde{A}_\alpha \end{aligned}\right\} \text{ DeMorgan's Laws}$$

SET CALCULUS

RECALL: Given two sets, A and B, we write:

- $A = B$ to mean $x \in A \iff x \in B$.
- $A \subseteq B$ to mean $x \in A \implies x \in B$.
- $A \cap B = \{x : x \in A \,\text{and}\, x \in B\}$. More generally, $\bigcap_{\alpha \in \Delta} A_\alpha = \{x : \forall \alpha \in \Delta,\, x \in A_\alpha\}$
- $A \cup B = \{x : x \in A \,\text{or}\, x \in B\}$. More generally, $\bigcup_{\alpha \in \Delta} A_\alpha = \{x : \exists \alpha \in \Delta,\, x \in A_\alpha\}$

To prove results about set operations, we'll make good use of propositional and predicate calculus!

EXAMPLE: Prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Sketch a corresponding Venn Diagram.

GOAL: We need to show that $x \in A \cap (B \cup C) \iff x \in (A \cap B) \cup (A \cap C)$.

EXAMPLE: Prove $A \cap \bigcup_{\alpha \in \Delta} B_\alpha = \bigcup_{\alpha \in \Delta} A \cap B_\alpha$.

GOAL:

EXAMPLE: Prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. Sketch a corresponding Venn Diagram.

GOAL:

EXAMPLE: Prove $A \cup \bigcap_{\alpha \in \Delta} B_\alpha = \bigcap_{\alpha \in \Delta} A \cup B_\alpha$.

GOAL:

RECALL: Given two sets A and B, the **set difference** $A \backslash B = A - B = \{x : x \in A \text{ and } x \notin B\}$

NOTE: If X is a universal set and $A \subseteq X$, then $X \backslash A = X - A = \tilde{A}$ is the **complement** of A.

EXAMPLE: Prove $\tilde{\tilde{A}} = A$.

EXAMPLE: If $A, B \subseteq X$, show $A - B = A \cap \tilde{B}$.

EXAMPLE: Prove DeMorgan's Laws for Sets: $\widetilde{A \cup B} = \tilde{A} \cap \tilde{B}$. Sketch a corresponding Venn Diagram.

EXAMPLE: Use the last two proofs to prove: $\widetilde{A \cap B} = \tilde{A} \cup \tilde{B}$

EXAMPLE: Prove $\widetilde{\bigcup\limits_{\alpha\in\Delta} A_\alpha} = \bigcap\limits_{\alpha\in\Delta} \widetilde{A_\alpha}$

EXAMPLE: Prove $\widetilde{\bigcap\limits_{\alpha\in\Delta} A_\alpha} = \bigcup\limits_{\alpha\in\Delta} \widetilde{A_\alpha}$

EXAMPLE: What is $\bigcup\limits_{\alpha\in\emptyset} A_\alpha$?

EXAMPLE: What is $\bigcap\limits_{\alpha\in\emptyset} A_\alpha$?